Cooling Coffee Without Solving Differential Equations

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"A mathematician is a device for turning coffee into theorems" —A. Rényi [3, p. 731]

Introduction

Almost all differential equations texts and many calculus texts have the following exercise: Assuming Newton's law of cooling, which strategy will yield a colder drink?

Be impatient: Add milk to hot coffee and wait five minutes.

Be patient: Wait five minutes, then add the cold milk.

Newton's law is that temperature $Y(t)$ obeys the differential equation $Y'(t) = -k\,(Y(t) - A)$, where A is room temperature; it is easy to find the explicit exponential solution and prove that the patient strategy is best. Of course, there are assumptions: the milk is kept cold during the time period, the mixing of the liquids is instantaneous, and the temperature of the mixed liquid becomes the weighted average of the temperatures of the constituents. We will say that the patient strategy *wins* when it yields the colder drink.

But for several reasons Newton's law is not an appropriate model. First, it is not an ironclad description of pure conduction. Over large temperature intervals it is not the case that a simple linear rate law governs transfer by conduction; models such as $Y^{5/4}$ (Lorentz law) or Y^p for other values of $p > 1$ have been proposed in other contexts. There appears to be no simple law of heat conduction that is generally applicable. More important for the coffee-and-milk problem are the other mechanisms of heat loss, such as radiation and evaporation. Experiments show that the loss through evaporation is quite large for water that starts near the boiling point. The trick of coating the liquid with oil will entirely remove evaporation, and so we can compare the effects experimentally as in Figure 1 (data generated by Macalester College students D. Nizam and R. Onkal).

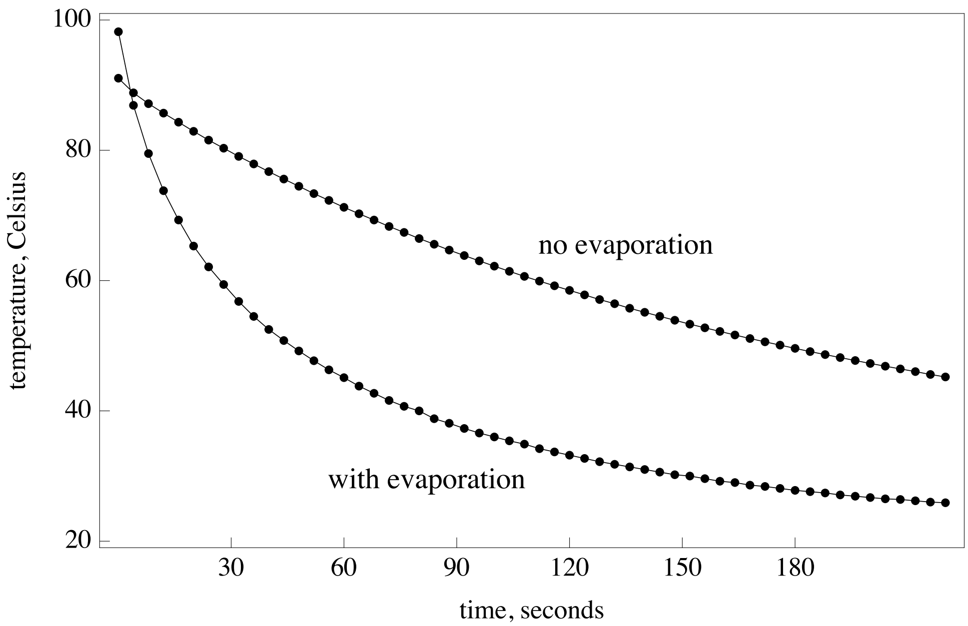

Figure 1. Cooling water, where an oil coating was used to suppress evaporation.

It is natural to ask if the problem of cooling coffee can be settled without having to set up and solve a differential equation that drives the cooling. Thus we want to find physically realistic conditions on $f(Y)$ that allow a mathematical proof that the patient strategy is best when heat loss is governed by the differential equation $Y' = -f(Y)$. We will prove here that the patient strategy wins for all parameter values if and only if $f(Y)$ "looks convex from the origin", a weakening of the usual definition of convexity that we formalize in the next section.

There are physical reasons why the patient strategy is best. If the milk were not kept cold but simply sat beside the coffee in an open container, then there should be no difference in the two methods. The change of temperature is roughly (not exactly: different heat coefficient, different evaporation effect) the same for both liquids and the result of mixing them is therefore roughly independent of the time at which they are mixed. But by staying cold until it is added, the milk is exempt from any heat absorption while the coffee cools; this exemption makes it seem reasonable that the patient strategy leads to a cooler mixed liquid.

However, note that, as observed in [2], there are two effects at work:

- arguing for patience: the coffee loses heat more quickly at the beginning, since under any reasonable model of heat loss, the loss is greater when the temperature difference is greater.

• arguing for impatience: if the cold milk is added to hot coffee, then the mixing reduces the temperature by more than if the milk is added to cooler coffee.

The interplay between the two mechanisms can be delicate. Consider the model $Y' = -\sqrt{Y}$, where the ambient temperature is 0, the coffee starts at 1, the milk is kept cold at -0.1, and the relative amount of coffee is 0.95. Figure 2 shows the two temperature profiles, with the patient drinker adding milk at 1 second. The impatient strategy yields the cooler coffee in this case.

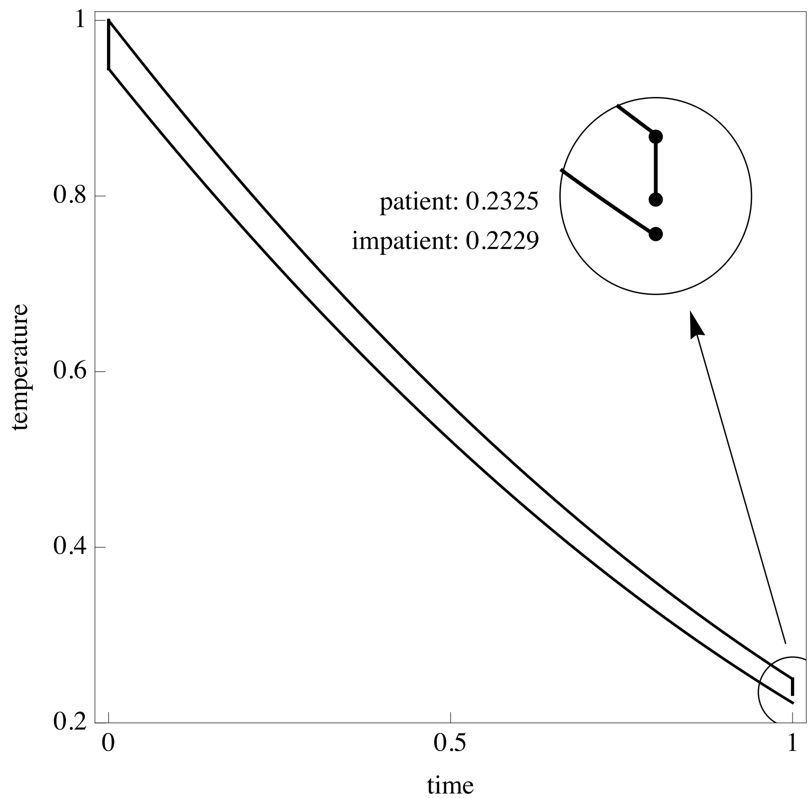

Figure 2. The impatient strategy leads to colder coffee when heat loss follows a square root law.

Other functions that arise in models of cooling are $Y' = A^4 - Y^4$ from the Stefan law of radiation (in the Kelvin scale) and an exponential function from an ODE approximation of evaporation. The Clausius–Clapeyron equation (see [1]) can be used to get such a model; it has the form (Kelvin scale): $Y' = c_0\left(e^{-c_1/Y} - 0.3\, e^{-c_1/A}\right)$, where 0.3 is a typical relative humidity, A is the ambient temperature (perhaps 293 K), and c_1 is derived from physical considerations such as the latent heat of evaporation and is about 4900. Modeling real data, using fine measurements of the mass lost to evaporation, led to a value for c_0 of 19000.

These various models are shown in Figure 3, where all functions have been rescaled to run from (0, 0) to (1, 1). The rescaling is for convenience, not for direct comparison: for hot coffee the evaporative loss is in fact greater than the losses from conduction or radiation. The figure includes a scaled version of $f(Y) = Y(\pi + 4\arctan(Y - 1))$, an interesting example that will be discussed later: the scaled version is nonconvex for $Y \geq 2/3$. There is a subtlety regarding evaporation, since that process continues and heat is lost even when the liquid is at the ambient temperature. As can be easily verified with two thermometers and a glass of water, the equilibrium temperature of water in a constant-temperature room is a little below room temperature.

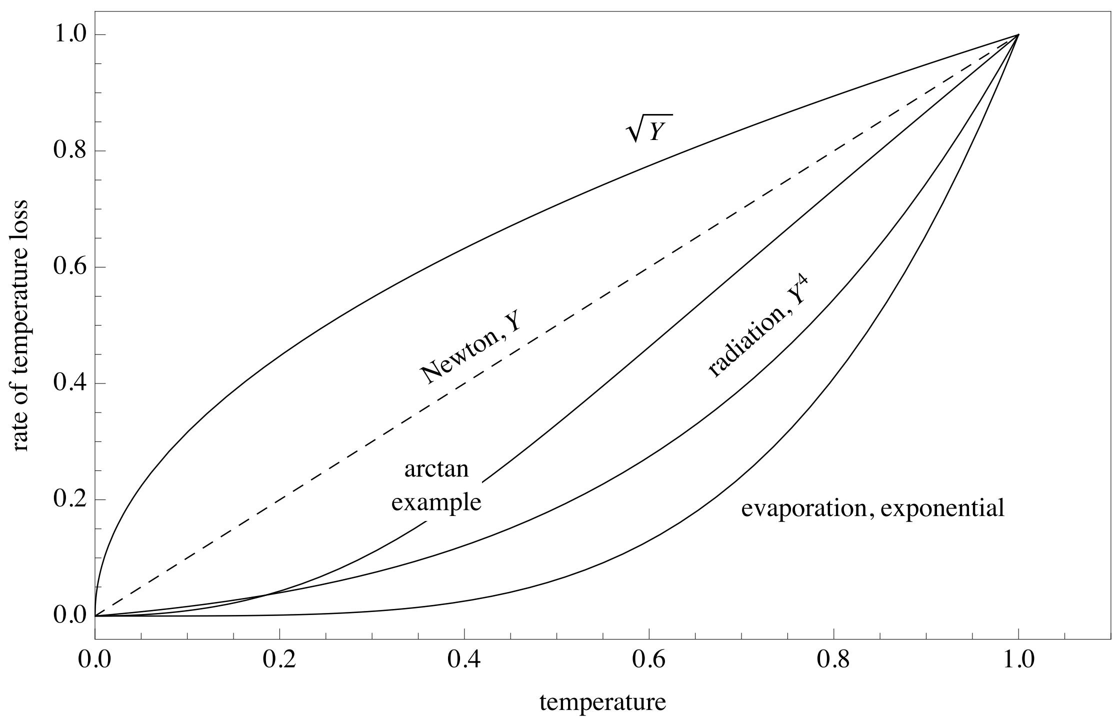

Figure 3. Various functions that arise in the study of cooling. Newton's law is the dashed linear function.

In the next section we will show that a certain convexity assumption on $f(Y)$ is the exact condition needed for the patient strategy to win.

When Patience Wins

Keeping to a scale where the equilibrium temperature is 0, it is clear that the temperature of real coffee must be nonincreasing, so we want $f(Y) \geq 0$ when $Y \geq 0$. The key to a general result about the patient strategy is the imposition of a weak form of convexity on $f(Y)$.

We focus here on continuously differentiable functions $f(Y)$ defined on $\mathbb{R}$, or possibly on $[0, \infty)$, with $f(0) = 0$ and $f(Y)$ positive (resp., negative) if Y is positive (resp., negative). Such a function is called a *cooling law*. We define a cooling law to be *V-convex* (resp., *V-concave*) if $f(Y)/Y$ is nondecreasing (resp., nonincreasing) on $(0, \infty)$. The "*V*" stands for *visually*, since, viewed from the origin, the graph is rising just as a convex function would. The physical interpretation of *V*-convexity is that the relative rate of heat loss is nondecreasing. This notion is weaker than convexity: an example (see Fig. 3) of a nonconvex cooling law that is *V*-convex is $Y(\pi + 4\arctan(Y - 1))$.

Note that a convex function ($f''(Y) \geq 0$) is *V*-convex (proof is an exercise using $f(0) = 0$ and three applications of the mean value theorem). The models in Figure 2, except for the square root function, are all *V*-convex; in fact, except for the arctan example, they are convex. Powers such as $\sqrt{Y}$ or $Y^{2/3}$ are concave and so not *V*-convex.

In what follows, C is the initial (positive) temperature of the coffee, M is the temperature of the milk (a negative constant), α (between 0 and 1) is the relative proportion of coffee, and $\beta = 1 - \alpha$. Thus impatient mixing gives the new temperature $\alpha C + \beta M$. We assume that the cooling process follows an unspecified differential equation $Y'(t) = -f(Y)$, where $Y(t)$ is the temperature of the liquid at time t. For convenience we assume that the equilibrium temperature is 0. We think of this as room temperature, but when evaporation is included (and the amount of water is not too small), the equilibrium value is a little less than room temperature.

Introduce F, the temperature function, by: $F(t, x) = Y(t)$ when the initial temperature is $Y(0) = x$. This assumes that f is continuously differentiable, so that the initial-value problem has a unique solution for all x. We will study in detail the cooling system $R_t(x)$, where $R_t(x) = F(t, x)$; it is often understood that t is fixed and then the subscript will be suppressed. It is a standard result that f being continuously differentiable implies the same for R. Now, the assertion that patience yields the cooler drink is the inequality $F(t, \alpha C + \beta M) > \alpha F(t, C) + \beta M$. We will show that this is always the case when $f(Y)$ is V-convex. Further, when f is not V-convex we can find values of the parameters where impatience yields the cooler drink.

Lemma 1. A function $f(Y)$ is V-convex iff $f'(Y) \geq f(Y)/Y$; it is V-concave iff $f'(Y) \leq f(Y)/Y$.

Proof. Apply the quotient rule. □

Lemma 2. For $t \geq 0$, $R_t'(x) = f(R_t(x))/f(x)$.

Proof. Because $f(0) = 0$, $Y = 0$ is an equilibrium point of the differential equation and it follows that $R_t(x) \geq 0$ whenever $x \geq 0$. Separate variables via $\int -1/f(Y)\, dY = t + c$ and let $g'(Y) = -1/f(Y)$; then $t + c = g(Y)$. Now assume initial temperature x and let $t = 0$ to obtain $c = g(x)$ and $t + g(x) = g(F(t, x))$. Let h be a local inverse of g. Then $F(t, x) = h(t + g(x))$ and

$$\frac{\partial}{\partial x} F(t, x) = \frac{g'(x)}{g'[h(t+g(x))]} = \frac{g'(x)}{g'[F(t,x)]} = \frac{f(F(t,x))}{f(x)} \quad \square$$

The next lemma shows how V-convexity flips between the cooling law and the temperature function.

Lemma 3. Let $f(Y)$ be a cooling law. Then $f(Y)$ is V-convex iff for all $t \geq 0$, $R_t(x)$ is a V-concave function of x.

Proof. A function f is V-convex iff for all $t \geq 0$, letting $R = R_t$, we have: $\frac{f(x)}{x} \geq \frac{f(R(x))}{R(x)}$ for nonnegative x (because $R(x) \leq x$ and $\lim_{x \to 0} R(x) = 0$). By Lemma 2, this is equivalent to $R(x)/x \geq f(R(x))/f(x) = R'(x)$, and Lemma 1 then concludes the proof. □

The next two theorems show that V-convexity is equivalent to the patient strategy winning for all parameter values.

Theorem 1. If a cooling law $f(Y)$ is V-convex, then the patient strategy always wins.

Proof. Assume α is large enough that $Y_0 = \alpha C + \beta M \geq 0$. For if it is not, then it is easy to see that the patient strategy wins. If α is such that Y_0 is negative, then the impatient temperature will only rise from Y_0, while the final mix for the patient drinker yields $\alpha Y(t) + \beta M \leq \alpha C + \beta M = Y_0$.

Fix t and use R for R_t. Let $h(y) = (R(\alpha C + y) - R(\alpha C))/y$ and define r to be $h(\beta M)$. The goal is: $R(\alpha C + \beta M) > \alpha R(C) + \beta M$. But the definitions of h and r mean that $R(\alpha C + \beta M) = R(\alpha C) + r \beta M$ and so it suffices to show (a) $R(\alpha C) \geq \alpha R(C)$, and (b)

$r < 1$. But $\alpha C < C$, so lemma 3 gives $\frac{R(\alpha C)}{\alpha C} \geq \frac{R(C)}{C}$, which yields (a). By the mean value theorem and lemmas 1 and 3, there is a $y \in [\alpha C + \beta M, \alpha C]$ such that $r = R'(y) \leq R(y)/y$. This last expression is less than 1 since temperature is strictly decreasing when it is positive, proving (b). □

Theorem 2. If a cooling law $f(Y)$ is not V-convex, then there are parameter values so that the patient strategy loses.

Proof. The hypothesis gives values C and C_1 such that $0 < C_1 < C$ and $f(C)/C < f(C_1)/C_1$. Because the temperature decreases to 0, there is $t > 0$ such that $R_t(C) = C_1$; fix this value of t. Then $f(C)/C < f(R(C))/R(C)$, or $R(C)/C < f(R(C))/f(C) = R'(C)$ by lemma 2. But $R(C)/C < R'(C)$ implies there's a $y < C$ with $R(y)/y < R(C)/C$ (choosing $y < C$ such that $\frac{R(C)-R(y)}{C-y} > \frac{R(C)}{C}$ works); letting $\alpha = y/C$, we then have $R(\alpha C)/(\alpha C) < R(C)/C$, or $R(\alpha C) < \alpha R(C)$. Now, as M approaches 0 from the left, $R(\alpha C + \beta M) - \beta M \to R(\alpha C) < \alpha R(C)$, so we can find $M < 0$ such that $R(\alpha C + \beta M) - \beta M < \alpha R(C)$. For these values of α, C, M, and t, the impatient strategy wins. □

To summarize, the cooling laws f for which patience wins are exactly those that arise by taking a C^1 function g that is defined on some interval of reals containing $[0, b)$, is nowhere negative, and is nondecreasing for nonnegative reals, and letting $f(Y) = Y\,g(Y)$. Newton's law is the case that g is constant.

Even when Theorem 2 applies, there can be values of the parameters for which the patient strategy wins. For example, if $f(Y) = \sqrt{Y}$ and t is the final time, it is easy to solve the differential equation algebraically and work out that the patient strategy wins whenever $16\,M \leq (1-\alpha)\,t^2 + 8\,\alpha\,t - 16\,\alpha$ (Fig. 4).

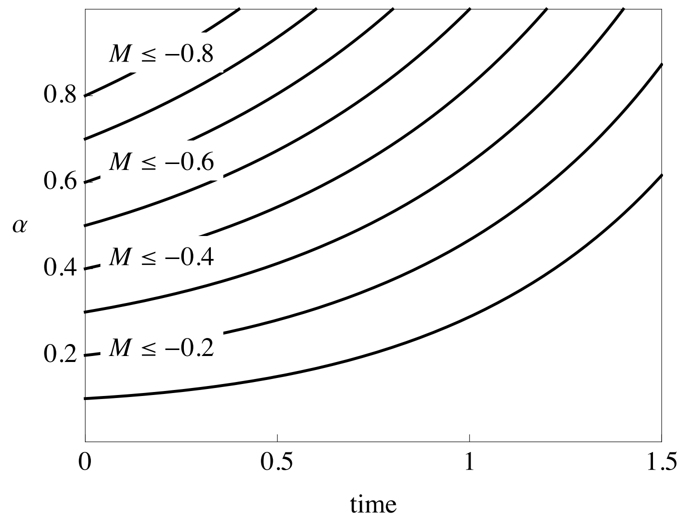

Figure 4. Upper bound on the milk temperature that allows the patient strategy to win for a square-root cooling law.

Temperature Half-Life

The preceding mathematical results are satisfying, but can we use physics to justify that a cooling law should be V-convex? One approach is to consider the concept of the half-life of the temperature: the amount of time needed for the temperature to drop to half its value (in the context of insulation of buildings, this is called the *time constant*: the time needed for the temperature to drop to $1/e$ of its value relative to ambient temperature). The half-life for a given starting value x is the time τ such that $F(x, \tau) = x/2$.

For Newton's law the half-life is constant, but that will not be the case in more general situations. It seems plausible that for any cooling law for water the half-life should be a nonincreasing function of x. Of course, this generalizes to ρ-life, where $0 < \rho < 1$. It turns out that the more general concept is equivalent to V-convexity. This is not surprising given that V-convexity is an assertion about relative loss of heat.

Theorem 3. A function has nonincreasing ρ-life for all ρ iff it is V-convex.

Proof. Using separation of variables, the ρ-life is $\int_{\rho x}^{x} \frac{1}{f(Y)}\, dY$. This being nonincreasing in x for $x > 0$ is equivalent to the derivative of the integral with respect to x being nonpositive: $\frac{1}{f(x)} - \frac{\rho}{f(\rho x)} \leq 0$, or $f(\rho x) \leq \rho f(x)$. This last being true for all $x > 0$ and $0 < \rho < 1$ is equivalent to $f(x)/x$ being nondecreasing for $x > 0$, the definition of V-

convexity. □

So all the models in Figure 3, except the square root, yield temperatures with decreasing ρ-life.

Conclusion

Our result assumed a first-order equation $Y'(t) = -f(Y)$ for the heat loss and showed that with a surprisingly wide class of cooling laws, patience always pays. A more exact treatment of heat transfer would assume that the temperature changes with position as well as time, thus leading to a partial differential equation. A PDE would also be relevant to a careful analysis of evaporation, since the humidity above the container would vary in a layered way. In some cases such PDEs might be well approximable by a first-order ODE, but that will not always be the case, especially for short time intervals. Thus we have the open problem of generalizing the results here to a PDE model of heat transfer.

Our work assumes that the coffee, with or without milk, obeys the same cooling law. This might not be entirely realistic, especially for radiation; the authors of [2] use Newton's law but with different coefficients for white vs. black coffee. Perhaps the ideas discussed here could be extended to such a two-model situation.

We gave a heuristic justification for V-convexity but one wonders whether there is a thermodynamic justification of this behavior for cooling liquids. It is tricky, since there are certain situations (e.g., the dropping of a red-hot metal block into cool water) where the cooling effect is not at all like cooling coffee: steam insulation can cause the cooling of the block to start off slowly, and later increase.

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SUMMARY FOR END OF PAPER. **A classic textbook problem is to show, assuming Newton's law of cooling, that if cold milk is added to coffee that has been cooling down, the result will be colder than if the milk was added at an earlier time. We formulate and prove a theorem that shows this holds when the linear function of Newton's law is replaced by any function satisfying a certain weak convexity condition. This is relevant to the real-world problem since Newton's law is not an adequate model for cooling liquids; it ignores the large amount of heat loss due to evaporation, as well as the smaller loss due to radiation.**